# A coherence test for variational preferences

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## Problem

- Under uncertainty, a set of $(\text{act}, \text{certainty equivalent})$ is given
  - measured in util or money
- When can the set be rationalized by a certain decision model?
  - expected utility
  - maxmin expected utility
  - variational preferences

Gilboa · Samuelson (2022) *TE*

New!

- Interpretations
  - Guide for coherence ← normative
  - Test for models in lab ← positive

# Overview

- Theorem of the alternative
  - *"Either ⟨model⟩-rationalizable or else ⟨condition⟩"*
  - expected utility ◦ maxmin expected utility 🆕 variational preferences
- Two different proofs

  1. Algebraic 2. Geometric

  - Differ from & simplify Gilboa · Samuelson's (2022)
- Clear geometric meaning of ⟨*condition*⟩
  - Closely related to the axioms for the models

## PRIMITIVES

- $\Omega = \{1, \ldots, W\}$: set of ***states***
- $\mathbb{T} = \{1, \ldots, T\}$: set of indices of acts

- $x^t \in \mathbb{R}^W$: ***act*** $t \in \mathbb{T}$ = mapping from states to payoffs
    - $x^t_\omega \in \mathbb{R}$: payoff of act $t$ at state $\omega$ monetary or utility unit
- $c^t \in \mathbb{R}$: ***certainty equivalent for*** $t$

When can $(x^t, c^t)_{t=1}^T$ be rationalized by a decision model?

# Rationalizability

## Definition

$(x^t, c^t)_{t=1}^T$ is …

- ***Bayesian-rationalizable*** if there exists $p \in \Delta$ s.t.

$$c^t = p^\intercal x^t \quad \forall t \in \mathbb{T}.$$

- ***maxmin-rationalizable*** if there exists a closed convex $P \subseteq \Delta$ s.t.

$$c^t = \min_{p \in P} p^\intercal x^t \quad \forall t \in \mathbb{T}.$$

- $\mathbf{1}_N \equiv (1, \ldots, 1)^\intercal \in \mathbb{R}^N$
- $\Delta \equiv \{ p \in \mathbb{R}_+^W \mid p^\intercal \mathbf{1}_W = 1 \}$

**Definition**

A ***cost function*** is $\gamma\colon \Delta \longrightarrow [0,\infty]$ s.t.

1. lower semicontinuous 2. convex 3. $\min \gamma(\Delta) = 0$.

**Definition** **cf. Maccheroni et al. (2006)**

$(x^t, c^t)_{t=1}^T$ is ***variational-rationalizable*** if there is a cost function $\gamma$ s.t.

$$c^t = \min_{p\in\Delta}(p^\top x^t + \gamma(p)) \quad \forall t \in \mathbb{T}.$$

# Variational model

## Example

Hansen · Sargent (2001)
Strzalecki (2011)

- Multiplier preference: for $(\theta, q) \in \mathbb{R}_{++} \times \Delta$,

$$\gamma(p) = \theta D_{\mathrm{KL}}(p \parallel q)$$

- Maxmin expected utility: for $P \subseteq \Delta$,

$$\gamma(p) = \begin{cases} 0 & \text{if } p \in P \\ \infty & \text{else} \end{cases}$$

◊

## Proposition

*Exactly one of the following holds.*

1. $(x^t, c^t)_{t=1}^T$ *is Bayesian-rationalizable.*
2. *There exists* $(\theta^t)_{t=1}^T \in \mathbb{R}^{\mathbb{T}}$ *s.t.*

$$\sum_{t=1}^{T} \theta^t c^t \mathbf{1}_W \gg \sum_{t=1}^{T} \theta^t x^t.$$

## BAYESIAN-RATIONALIZABILITY

### Proposition

*Either* 1. $\exists p \in \Delta \ \forall t \in \mathbb{T} \ \ c^t = p^\top x^t$ *or else*

2. $\exists \theta \in \mathbb{R}^{\mathbb{T}} \ \sum_{t=1}^{T} \theta^t c^t \mathbf{1}_W \gg \sum_{t=1}^{T} \theta^t x^t.$

Mathematically, ...

- identical to the fundamental theorem of finance
  - *"Either $\exists$ risk-neutral prob. or else $\exists$ arbitrage opportunity"*
- equivalent to Farkas' lemma

# Maxmin-rationalizability

**Proposition** © Gilboa · Samuelson (2022)

*Exactly one of the following holds.*

1. $(x^t, c^t)_{t=1}^T$ *is maxmin-rationalizable.*
2. *There exists* $(\theta^t)_{t=1}^T \in \mathbb{R}^{\mathbb{T}}$ *s.t.*

$$\sum_{t=1}^T \theta^t c^t \mathbf{1}_W \gg \sum_{t=1}^T \theta^t x^t, \qquad |\mathbb{T}_-(\theta)| \leq 1.$$

- $\mathbb{T}_-(\theta) \equiv \{ t \in \mathbb{T} \mid \theta^t < 0 \}$

## Theorem

*Exactly one of the following holds.*

1. $(x^t, c^t)_{t=1}^T$ *is variational-rationalizable.*
2. *There exists* $(\theta^t)_{t=1}^T \in \mathbb{R}^{\mathbb{T}}$ *s.t.*

$$\sum_{t=1}^T \theta^t c^t \mathbf{1}_W \gg \sum_{t=1}^T \theta^t x^t, \qquad \begin{cases} \mathbb{T}_-(\theta) = \emptyset & \text{or else} \\ |\mathbb{T}_-(\theta)| = 1 \;\; \& \;\; \sum_{t=1}^T \theta^t \leq 0. \end{cases}$$

- $\mathbb{T}_-(\theta) \equiv \{\, t \in \mathbb{T} \mid \theta^t < 0 \,\}$

## Remarks

**Theorem**

*Either* 1. $(x^t, c^t)_{t=1}^T$ *is variational-rationalizable* *or else*

2. *there exists* $(\theta^t)_{t=1}^T \in \mathbb{R}^{\mathbb{T}}$ *s.t.* $\sum_{t=1}^T \theta^t c^t \mathbf{1}_W \gg \sum_{t=1}^T \theta^t x^t$ *and*

$$\mathbb{T}_-(\theta) = \emptyset \qquad \text{or} \qquad |\mathbb{T}_-(\theta)| = 1 \;\;\&\;\; \sum_{t=1}^T \theta^t \le 0.$$

1. $\dim(\text{set of cost functions}) = \infty$
2. System of finite-dimensional linear inequalities
   - Interpretation $\leftsquigarrow$ geometric proof

**Theorem**

*Either* 1. $(x^t, c^t)_{t=1}^T$ *is variational-rationalizable* *or else*

2. *there exists* $(\theta^t)_{t=\mathbf{0}}^T \in \mathbb{R}^{\hat{\mathbb{T}}}$ *s.t.* $\sum_{t=\mathbf{0}}^T \theta^t c^t \mathbf{1}_W \gg \sum_{t=\mathbf{0}}^T \theta^t x^t$ *and*

$$\hat{\mathbb{T}} \equiv \{0, 1, \ldots, T\}, \qquad |\hat{\mathbb{T}}_-(\theta)| = 1 \;\; \& \;\; \sum_{t=\mathbf{0}}^T \theta^t \mathbf{=} 0.$$

- For each of the models,

$$(x^t, c^t)_{t=1}^T \text{ is rationalizable } \iff (x^t, c^t)_{t=\mathbf{0}}^T \text{ is rationalizable}$$

where $(x^0, c^0) \equiv (0, 0)$ $\qquad c^0 = 0 = \min_{p\in\Delta} \gamma(p) = \min_{p\in\Delta}(p^\top x^0 + \gamma(p))$

# Algebraic proof—sketch

## Claim

$(x^t, c^t)_{t=1}^T$ *is variational-rationalizable* $\iff$

$\langle\star\rangle$ *for each* $t \in \hat{\mathbb{T}}$, *there exists* $(q^t, r^t) \in \Delta \times \mathbb{R}_+$ *s.t.*

$$(q^t)^\top x^u + r^t \geq c^u \quad \forall u \in \hat{\mathbb{T}}, \qquad (q^t)^\top x^t + r^t = c^t.$$

## Claim

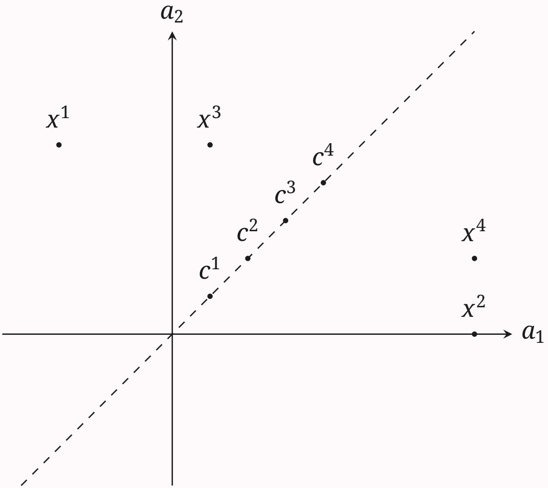
**A variant of Farkas' lemma**

*Either* $\langle\star\rangle$ *or else* $\langle\spadesuit\rangle$ *there exists* $(t, (\theta^u)_{u=0}^T) \in \hat{\mathbb{T}} \times \mathbb{R}^{\hat{\mathbb{T}}}$ *s.t.*

$$\sum_{u=0}^T \theta^u c^u \mathbf{1}_W \gg \sum_{u=0}^T \theta^u x^u, \qquad \sum_{u=0}^T \theta^u \leq 0, \qquad \theta^u \geq 0 \quad \forall u \in \hat{\mathbb{T}} \setminus \{t\}.$$

- Variational-rationalizability $\iff$ $\langle\star\rangle$ $\iff$ $\neg\langle\spadesuit\rangle$

# Geometric proof




$$W = 2, \qquad T = 4, \qquad (x^t, c^t)_{t=1}^{T}$$

## Geometric proof—preliminaries

### Definition

A function $V \colon \mathbb{R}^W \to \mathbb{R}$ is

- ***normalized*** if

$$V(r\mathbf{1}_W) = r \quad \forall r \in \mathbb{R};$$

- ***monotone*** if

$$a \geq b \implies V(a) \geq V(b);$$

- ***translation equivariant*** if

$$V(a + r\mathbf{1}_W) = V(a) + r \quad \forall (a, r) \in \mathbb{R}^W \times \mathbb{R}.$$

**Proposition** **© Maccheroni et al. (2006)**

$\langle\diamond\rangle$ *$V$ is normalized + monotone + translation equivariant + quasiconcave*

$\Longleftrightarrow$ *there exists a cost function $\gamma$ s.t.*

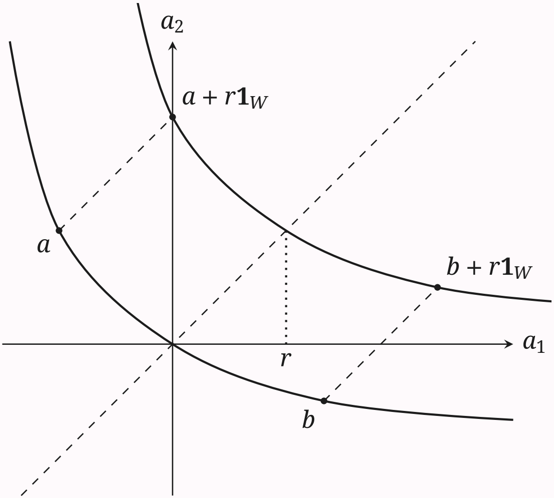
$$V(a) = \min_{p \in \Delta} (p^\top a + \gamma(p)) \quad \forall a \in \mathbb{R}^W.$$

- $\mathscr{V} \equiv \{ V : \mathbb{R}^W \to \mathbb{R} \mid \langle\diamond\rangle \}$

$(x^t, c^t)_{t=1}^T$ is variational-rationalizable

$$\Longleftrightarrow \exists V \in \mathscr{V} \quad \forall t \in \mathbb{T} \quad c^t = V(x^t)$$

## Geometric proof



**Lemma**

*For each* $t \in \hat{\mathbb{T}}$, *let* $(\tilde{x}^t, \tilde{c}^t) \equiv (x^t - c^t \mathbf{1}_W, 0)$. *Then,*

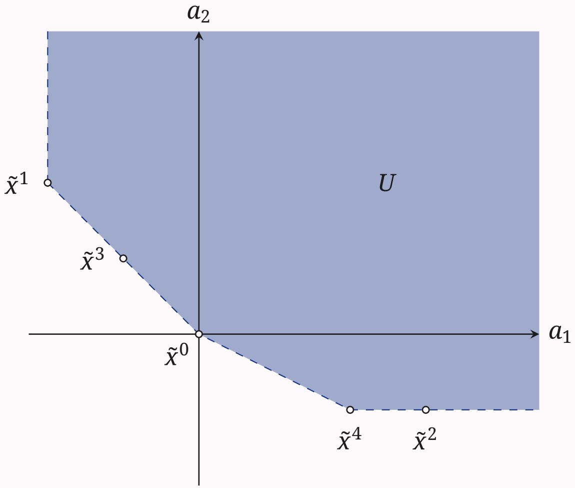
$$(x^t, c^t)_{t=1}^T \text{ is variational-rationalizable}$$

$$\iff (\tilde{\boldsymbol{x}}^{\boldsymbol{t}}, \tilde{\boldsymbol{c}}^{\boldsymbol{t}})_{t=\mathbf{0}}^T \text{ is variational-rationalizable.}$$

- If $V$ is translation equivariant, then for each $t \in \hat{\mathbb{T}}$,

$$c^t = V(x^t) \iff 0 = V(x^t - c^t \mathbf{1}_W) = V(\tilde{x}^t) \qquad \square$$

# Geometric proof




$$I \equiv \{\, \tilde{x}^t \mid t \in \hat{\mathbb{T}} \,\}, \qquad U \equiv \operatorname{co}(I) + \mathbb{R}^W_{++}, \qquad I \cap U =\neq \emptyset$$

### Claim

*Let* $I \equiv \{\, \tilde{x}^t \mid t \in \hat{\mathbb{T}} \,\}$ *and* $U \equiv \mathrm{co}(I) + \mathbb{R}^W_{++}$. *Then,*

$$(x^t, c^t)_{t=1}^T \text{ is variational-rationalizable} \iff I \cap U = \emptyset.$$

## Geometric proof

### Theorem

*Either* 1. $(x^t, c^t)_{t=1}^T$ *is variational-rationalizable* *or else*

2. *there exists* $(\theta^t)_{t=0}^T \in \mathbb{R}^{\hat{\mathbb{T}}}$ *s.t.* $\sum_{t=0}^T \theta^t c^t \mathbf{1}_W \gg \sum_{t=0}^T \theta^t x^t$ *and*

$$|\hat{\mathbb{T}}_-(\theta)| = 1 \quad \& \quad \sum_{t=0}^T \theta^t = 0.$$

$$
\begin{aligned}
I \cap U \neq \emptyset &\iff \exists (t, a) \in \hat{\mathbb{T}} \times \mathrm{co}(I) \quad \tilde{x}^t \gg a \\
&\iff \exists (t, \lambda) \in \hat{\mathbb{T}} \times \mathbb{R}_+^{\hat{\mathbb{T}}} \qquad \tilde{x}^t \gg \sum_{u=0}^T \lambda^u \tilde{x}^u \quad \sum_{u=0}^T \lambda^u = 1 \\
&\iff \exists (t, \theta) \in \hat{\mathbb{T}} \times \mathbb{R}^{\hat{\mathbb{T}}} \qquad \begin{cases} \sum_{u=0}^T \theta^u c^u \mathbf{1}_W \gg \sum_{u=0}^T \theta^u x^u \\ \sum_{u=0}^T \theta^u = 0, \quad |\hat{\mathbb{T}}_-(\theta)| = 1 \end{cases} \qquad \square
\end{aligned}
$$

**Definition**

A function $V : \mathbb{R}^W \to \mathbb{R}$ is ***positively homogeneous*** if

$$V(\alpha a) = \alpha V(a) \quad \forall (\alpha, a) \in \mathbb{R}_+ \times \mathbb{R}^W.$$

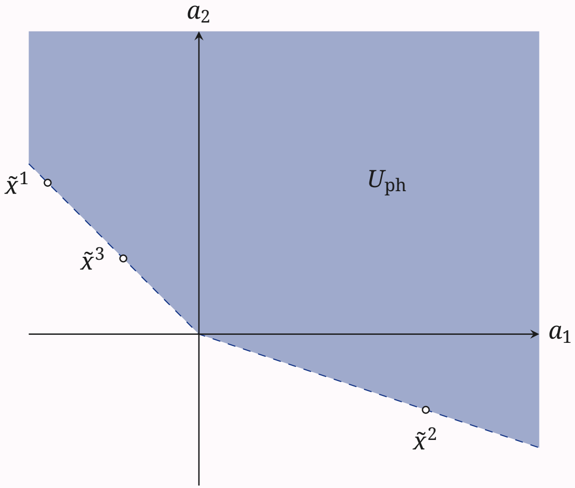
**Proposition** **© Gilboa · Schmeidler (1989)**

*$V \in \mathscr{V}$ is positively homogeneous* $\iff$

*there exists a closed convex $P \subseteq \Delta$ s.t.*

$$V(a) = \min_{p \in P} p^\top a \quad \forall a \in \mathbb{R}^W.$$

# Maxmin-rationalizability




$$I = \{\, \tilde{x}^t \mid t \in \mathbb{T} \,\}, \qquad U_{\text{ph}} = \mathbf{coni}(I) + \mathbb{R}^W_{++}, \qquad I \cap U_{\text{ph}} = \emptyset$$

## Maxmin-rationalizability

**Proposition** **© Gilboa · Samuelson (2022)**

*Either* 1. $(x^t, c^t)_{t=1}^T$ *is maxmin-rationalizable or else*

2. *there exists* $(\theta^t)_{t=1}^T \in \mathbb{R}^{\mathbb{T}}$ *s.t.* $\sum_{t=1}^T \theta^t c^t \mathbf{1}_W \gg \sum_{t=1}^T \theta^t x^t$ *and*

$$|\mathbb{T}_-(\theta)| \leq 1.$$

$$I \cap U_{\text{ph}} \neq \emptyset \iff \exists (t, \alpha) \in \mathbb{T} \times \mathbb{R}_+^{\mathbb{T}} \quad \tilde{x}^t \gg \sum_{u=1}^T \alpha^u \tilde{x}^u$$

$$\iff \exists (t, \theta) \in \mathbb{T} \times \mathbb{R}^{\mathbb{T}} \quad \begin{cases} \sum_{u=1}^T \theta^u c^u \mathbf{1}_W \gg \sum_{u=1}^T \theta^u x^u \\ |\mathbb{T}_-(\theta)| \leq 1 \end{cases} \qquad \square$$

## Geometric proof—comments

- $(x^t, c^t)_{t=1}^T$ *is Bayesian / maxmin / variational-rationalizable* $\Longleftrightarrow$ $I$ *does not meet* $\mathrm{span}(I) + \mathbb{R}^W_{++}$ / $\mathrm{coni}(I) + \mathbb{R}^W_{++}$ / $\mathrm{co}(I) + \mathbb{R}^W_{++}$
  - Unified viewpoint for the dual conditions
- Separation argument is not directly used — e.g., Farkas' lemma
  - Implicit in the characterization of preference functionals
- Non-identification
  - …but the most ambiguity-averse one is constructed — for MM & VP
- Finiteness of $\Omega$ & $\mathbb{T}$ is not used
  - ➜ The same argument works for infinite $\Omega$ and/or $\mathbb{T}$
  - In contrast to the algebraic proof

## Summary

- When can a set of $(\text{act}, \text{certainty equivalent})$ be rationalized?
  - *Either ⟨model⟩-rationalizable or else ⟨condition⟩*
  - expected utility ◦ maxmin expected utility 🆕 variational preferences

- Two simple proofs: algebraic & geometric
- Clear geometric meaning of ⟨*condition*⟩
  - Shape of the upper contour set generated by translated acts

# Appendix

A Algebraic proof

B General $\Omega$ & $\mathbb{T}$

## Algebraic proof—notation

$$\underbrace{X}_{W\times T} \equiv \begin{bmatrix} x_1^1 & \cdots & x_1^T \\ \vdots & \ddots & \vdots \\ x_W^1 & \cdots & x_W^T \end{bmatrix}, \qquad c \equiv \begin{bmatrix} c^1 \\ \vdots \\ c^T \end{bmatrix}, \qquad \underbrace{D^t}_{T\times T} \equiv \operatorname{diag}(e^t - \mathbf{1}_T),$$

$$\operatorname{diag}(v) = \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{bmatrix}, \qquad e^0 = 0_T, \qquad e_u^t = \begin{cases} 1 & \text{if } u = t \\ 0 & \text{else} \end{cases}$$

## Claim

$(x^t, c^t)_{t=1}^T$ *is variational-rationalizable* $\Longleftrightarrow$

$\langle\star\rangle$ *for each* $t \in \hat{\mathbb{T}}$*, there exists* $(q^t, r^t, s^t) \in \Delta \times \mathbb{R}_+ \times \mathbb{R}_+^T$ *s.t.*

$$(q^t)^\top (X - \mathbf{1}_W c^\top) + r^t \|e^t\| \mathbf{1}_T^\top + (s^t)^\top D^t = 0_T^\top.$$

### Motzkin transposition theorem

*Let* $(L, M) \in \mathbb{R}^{\ell \times n} \times \mathbb{R}^{m \times n}$*. Exactly one of the following holds.*

1. *There exists* $z \in \mathbb{R}^n$ *s.t.* $Lz \ll 0_\ell$ *and* $Mz \leq 0_m$*.*
2. *There exists* $(u, v) \in \mathbb{R}^\ell_+ \times \mathbb{R}^m_+$ *s.t.* $u^\top L + v^\top M = 0_n^\top$ *and* $u^\top \mathbf{1}_\ell = 1$*.*

- By the Motzkin transposition theorem, either

  $\langle \star \rangle$ for each $t \in \hat{\mathbb{T}}$, there exists $(q^t, r^t, s^t) \in \Delta \times \mathbb{R}_+ \times \mathbb{R}^T_+$ s.t.

  $$(q^t)^\top (X - \mathbf{1}_W c^\top) + r^t \|e^t\| \mathbf{1}_T^\top + (s^t)^\top D^t = 0_T^\top \qquad \text{or else}$$

  $\langle \spadesuit \rangle$ there exists $(t, \theta) \in \hat{\mathbb{T}} \times \mathbb{R}^T$ s.t.

  $$\sum_{u=1}^T \theta^u x^u \ll \sum_{u=1}^T \theta^u c^u \mathbf{1}_W, \qquad \|e^t\| \sum_{u=1}^T \theta^u \leq 0, \qquad \theta^u \geq 0 \quad \forall u \neq t \qquad \square$$

# Appendix

A Algebraic proof

B General $\Omega$ & $\mathbb{T}$

# General $\Omega$ & $\mathbb{T}$

- $\Omega$: set of ***states*** • $\mathscr{E}$: algebra of ***events*** • $\mathbb{T}$: set of indices of acts
- $(x^t, c^t)_{t\in\mathbb{T}} \in (\mathrm{B_b}(\mathscr{E}) \times \mathbb{R})^{\mathbb{T}}$: act–certainty-equivalent pairs
    - $\mathrm{B_b}(\mathscr{E})$: set of bounded $\mathscr{E}$-measurable functions
- $\Delta$: set of (finitely additive) probabilities on $\mathscr{E}$

**Definition**

$(x^t, c^t)_{t\in\mathbb{T}}$ is ***variational-rationalizable*** if there is a cost function $\gamma$ s.t.

$$c^t = \min_{p\in\Delta}\left(\int x^t \,\mathrm{d}p + \gamma(p)\right) \quad \forall t \in \mathbb{T}.$$

**Theorem**

*Exactly one of the following holds.*

1. $(x^t, c^t)_{t\in\mathbb{T}}$ *is variational-rationalizable.*
2. *There exists* $\theta \in \mathbb{R}^{(\mathbb{T})}$ *s.t.*

$$\sum_{t\in\mathbb{T}} \theta(t) c^t \mathbf{1}_\Omega \ggg \sum_{t\in\mathbb{T}} \theta(t) x^t, \qquad \begin{cases} \mathbb{T}_-(\theta) = \emptyset & \text{or else} \\ |\mathbb{T}_-(\theta)| = 1 \quad \& \quad \sum_{t\in\mathbb{T}} \theta(t) \leq 0. \end{cases}$$

- $\mathbb{R}^{(\mathbb{T})} \equiv \{\, \theta \in \mathbb{R}^{\mathbb{T}} \mid |\{\, t \in \mathbb{T} \mid \theta(t) \neq 0 \,\}| < \infty \,\}$
- $a \ggg b \overset{\text{def}}{\iff} \inf_{\omega\in\Omega}(a-b)(\omega) > 0$