A simple proof of the representation theorem for betweenness preferences

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This paper presents a simple proof of Dekel (1986)'s representation theorem for betweenness preferences. The proof is based on the separation theorem.

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1. Introduction

In non-expected utility theory under risk, Dekel (1986) characterizes the class of preferences satisfying the *betweenness* axiom. He shows that those preferences have implicit expected utility representations. The representation theorem is generalized by Payró (2023) to preferences over menus of lotteries, and is applied to a model of temptation and self-control.

Their proofs, however, are long and involved. This paper presents a shorter and more intuitive proof. I show the representation result for preferences on a compact convex subset of a topological vector space. This formulation is general enough to cover the baseline settings of Dekel (1986) and Payró (2023).[1] My proof is based on the separation theorem: the upper and lower contour sets of the betweenness preference are convex and separated by an affine functional, which identifies the local utility index. However, the direct application of this argument may fail when the domain has infinite dimensions. For example, the space of Borel probability measures on $[0,1]$ endowed with the weak$*$ topology has empty relative interior, so the separation theorem is inapplicable. To overcome this difficulty, I employ a technique similar to that of Chatterjee and Krishna (2008): I first apply the separation theorem to get the local utility indices on finite-dimensional subsets and then "concatenate" those indices to construct the index on the whole domain.

Another attempt to simplify the original proof has been made by Conlon (1995) in a setting of preferences over lotteries. Instead of the separation theorem, applying the convexity of the indifference sets, he obtains hyperplanes to construct the local utility indices. His argument relies on a geometric structure of the probability simplex: the local utility is assigned to each lottery using the hyperplane and the line passing through the lottery and the origin in the

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[1] Payró (2023) also considers a more general setting, which requires a more complicated proof.

vector space spanned by all the lotteries. In contrast, my construction exploits only the affine structure of the domain and thus is independent of the zero in the vector space.

2. Model

Let $\succsim$ be a binary relation on a nonempty compact convex subset X of a Hausdorff topological vector space. Denote by $\sim$ and $\succ$ the symmetric and asymmetric parts of $\succsim$, respectively.

The following axioms are standard.

Axiom 1 (Rationality)**.** The relation $\succsim$ is complete and transitive.

Axiom 2 (Nondegeneracy)**.** There exists $(x, y) \in X^2$ such that $x \succ y$.

Axiom 3 (Continuity)**.** The upper and lower contour sets of $\succsim$ are closed everywhere.

The key axiom is the following, which is studied by Dekel (1986) and Chew (1989).

Axiom 4 (Betweenness)**.** For each $(\lambda, (x, y)) \in (0, 1) \times X^2$, if $x \succ y$, then $x \succ \lambda x + (1 - \lambda) y \succ y$.

Together with the previous axioms, *betweenness* follows in the weak preference form.

Lemma 1. *If $\succsim$ satisfies rationality, continuity, and betweenness, then $x \succsim y$ implies $x \succsim \lambda x + (1 - \lambda) y \succsim y$ for each $(\lambda, (x, y)) \in (0, 1) \times X^2$.*

Proof. Assume the axioms. Choose any $(\lambda, (x, y)) \in (0, 1) \times X^2$ with $x \succsim y$. Suppose that there exists $z \in X$ such that $y \succ z$. Otherwise, the similar argument works by taking $z \in X$ with $z \succ x$. Define the sequence $(w_n)_{n\in\mathbb{N}}$ by $w_n = (1 - n^{-1}) y + n^{-1} z$, which converges to y. For each $n \in \mathbb{N}$, since $y \succ w_n$ by *betweenness*, it follows from *rationality* that $x \succ w_n$, so $x \succ \lambda x + (1 - \lambda) w_n \succ w_n$ by *betweenness*. Thus, *continuity* implies $x \succsim \lambda x + (1 - \lambda) y \succsim y$. □

A real-valued function f on X is **mixture linear** if $f(\lambda x + (1 - \lambda) y) = \lambda f(x) + (1 - \lambda) f(y)$ for each $(\lambda, (x, y)) \in [0, 1] \times X^2$; a real-valued function U on X **represents** $\succsim$ if $x \succsim y$ is equivalent to $U(x) \geq U(y)$. An **implicit mixture linear representation of** $\succsim$ is a real-valued function u on $X \times [0, 1]$ such that

(i) $u(\cdot, t)$ is continuous and mixture linear for each $t \in (0, 1)$;

(ii) $u(x, \cdot)$ is continuous on $(0, 1)$ for each $x \in X$;

(iii) there exists $(x^*, x_*) \in X^2$ such that $u(x^*, t) = 1$ and $u(x_*, t) = 0$ for each $t \in [0, 1]$;

(iv) for each $x \in X$, there exists unique $t \in [0, 1]$ such that $t = u(x, t)$;

(v) the continuous real-valued function U on X of the form $U(x) = u(x, U(x))$ represents $\succsim$.

In the rest of the paper, I prove the following theorem.

Theorem 1 (Dekel, 1986; Payró, 2023)**.** *The relation $\succsim$ satisfies rationality, nondegeneracy, continuity, and betweenness if and only if it admits an implicit mixture linear representation.*

3. Proof of the theorem

The necessity of the axioms is routine. For the sufficiency, assume all the axioms.

By *rationality*, *continuity*, and the compactness of X, there exists $(x^*, x_*) \in X^2$ such that $x^* \succeq x \succeq x_*$ for each $x \in X$. By *nondegeneracy*, $x^* \succ x_*$. For each $t \in [0,1]$, let $m_t = tx^* + (1-t)x_*$. For each $(x,t) \in X \times (0,1)$, let $\xi_t(x) = x_*$ if $x \succeq m_t$ and let $\xi_t(x) = x^*$ otherwise. By *rationality*, *continuity*, and *betweenness*, there exist unique functions $U\colon X \to [0,1]$ and $\mu\colon X \times (0,1) \to (0,1]$ such that $x \sim m_{U(x)}$ for each $x \in X$ and $m_t \sim \mu(x,t)x + (1-\mu(x,t))\xi_t(x)$ for each $(x,t) \in X \times (0,1)$, which are continuous by *continuity*. By *betweenness*, U represents $\succeq$.

Let Γ be the collection of all finite-dimensional convex subsets of X that include $\{x^*, x_*\}$. For each $(t, C) \in (0,1) \times \Gamma$, since by *rationality* and Lemma 1, $\{\, x \in C \mid x \succeq m_t \,\}$ and $\{\, x \in C \mid m_t \succ x \,\}$ are convex, the separation theorem (Rockafellar, 1970, Theorem 11.3) and *continuity* imply that there exists an affine functional v_t^C on the affine hull of C such that for each $x \in C$,

$$x \succeq m_t \iff v_t^C(x) \geq v_t^C(m_t), \qquad x \succ m_t \iff v_t^C(x) > v_t^C(m_t).$$

Passing to a normalization, assume $v_t^C(x^*) = 1$ and $v_t^C(x_*) = 0$. Then, for each $(x,t) \in X \times (0,1)$, the value $v_t^C(x)$ is independent of the choice of C as long as $x \in C$: for each $C \in \Gamma$ with $x \in C$, it follows that $t = v_t^C(m_t) = \mu(x,t)v_t^C(x) + (1-\mu(x,t))v_t^C(\xi_t(x))$, so

$$v_t^C(x) = \frac{t - (1-\mu(x,t))v_t^C(\xi_t(x))}{\mu(x,t)} = \begin{cases} \dfrac{t}{\mu(x,t)} & \text{if } U(x) \geq t, \\ 1 - \dfrac{1-t}{\mu(x,t)} & \text{otherwise.} \end{cases} \tag{1}$$

For each $x \in X$, let $H(x)$ be the convex hull of $\{x^*, x_*, x\}$ so that $v_t^C(x) = v_t^{H(x)}(x)$ for each $(t, C) \in (0,1) \times \Gamma$ with $x \in C$. For each $z \in \{x^*, x_*\}$, let $I(z) = \{\, x \in X \mid x \sim z \,\}$. Denote by $\mathbf{1}_A$ the indicator function of A on X. Define $u\colon X \times [0,1] \to \mathbb{R}$ by

$$u(x,t) = \begin{cases} \mathbf{1}_{X \setminus I(x_*)}(x) & \text{if } t = 0, \\ v_t^{H(x)}(x) & \text{if } t \in (0,1), \\ \mathbf{1}_{I(x^*)}(x) & \text{if } t = 1. \end{cases}$$

It remains to show that u is an implicit mixture linear representation of $\succeq$. For each $t \in (0,1)$, since for each $(\lambda, (x,y)) \in [0,1] \times X^2$, letting C be the convex hull of $\{x^*, x_*, x, y\}$ gives

$$u(\lambda x + (1-\lambda)y, t) = v_t^C(\lambda x + (1-\lambda)y) = \lambda v_t^C(x) + (1-\lambda)v_t^C(y) = \lambda u(x,t) + (1-\lambda)u(y,t),$$

the function $u(\cdot, t)$ is mixture linear. From (1) and the continuity of μ, the function u is continuous on $X \times (0,1)$. By construction, $u(x^*, t) = 1$ and $u(x_*, t) = 0$ for each $t \in [0,1]$. Finally, $1 = u(x,1)$ if and only if $x \sim x^*$; $0 = u(x,0)$ if and only if $x \sim x_*$; for each $t \in (0,1)$, it follows that $t = u(x,t)$ if and only if $t = v_t^{H(x)}(x)$, which is equivalent to $t = U(x)$. □

References

Chatterjee, K. and R. V. Krishna (2008). "A geometric approach to continuous expected utility." *Economics Letters*, 98, 89–94. [1]

Chew, S. H. (1989). "Axiomatic utility theories with the betweenness property." *Annals of Operations Research*, 19, 273–298. [2]

Conlon, J. R. (1995). "A simple proof of a basic result in nonexpected utility theory." *Journal of Economic Theory*, 65, 635–639. [1]

Dekel, E. (1986). "An axiomatic characterization of preferences under uncertainty: Weakening the independence axiom." *Journal of Economic Theory*, 40, 304–318. [1, 2]

Payró, F. (2023). "Mixture-dependent prefernece for commitment." Working paper. [1, 2]

Rockafellar, T. R. (1970). *Convex Analysis*: Princeton University Press. [3]